

# Response–Correlation Inequality in Dynamical Systems

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The flurry of activity in non equilibrium statistical phenomena covers many fields of theoretical and practical importance such as growth models, front propagation, crack propagation and many more. In this article we derive an exact inequality relating the response function, measuring the steady state response of the physical field of interest to an external probe, and the correlation function. With no further assumptions, that inequality is turned into an exponent inequality. For growth models of the general KPZ family at steady state, the response and dynamic exponents are equal. This leads to a more stringent inequality that relates the roughness and growth exponents. To demonstrate the usefulness of the inequality, we consider the NKPZ family of growth models and show that various approaches to that problem produce exponents, which violate the inequality. We also consider some experimental and numerical results that also seem to violate the inequality and suggest measurements that would clarify the origin of the difficulty.

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The focus of interest in statistical physics has shifted in the last two decades from equilibrium phase transitions and later the dynamics of phase transitions [1] to the study of non-equilibrium systems such as various growth models [2, 3, 4, 5, 6], front propagations [6, 7], crack propagation [8, 9, 10, 11] etc. In spite of that shift the main objects of study remained of a similar nature, a small set of exponents which describe the steady state properties as well as the evolution of the system. Except for a number of one dimensional exactly soluble problems [2, 3, 6, 12, 13, 14, 15], the sets of exponents given in the literature for many systems belonging to the above categories, vary considerably from author to author and depend strongly on the method of derivation. This is very different from the situation in equilibrium phase transitions, where methods as different as high temperature expansion, momentum space RG and real space RG yield very close exponents. Under such circumstances rigorous results that can put bounds on the exponents describing the system are obviously most valuable. In the following we present a quite powerful inequality for dynamical stochastic systems, which is an extension of the Schwartz-Soffer inequality derived for quenched random problems [16]. The inequality is of a generic nature and relates the response at steady state of some measurable physical field to an external disturbance to the time dependent correlations of that physical field. This in itself is enough to check approximation schemes or experiments that supply both quantities. In the interesting cases, where the system may be described in terms of set of exponents, the predictions of the inequality become

more dramatic by turning it into an exponent inequality. In order to assess the usefulness of the exponent inequality, we apply it, as an example, to the results obtained by different methods for one specific model, the non-local Kardar-Parisi-Zhang (KPZ) equation [4]. This test demonstrates not only the obvious discrepancies among results obtained by different methods, as discussed above but also that all methods but one, namely the self consistent expansion (SCE) [17, 18, 19], violate the inequality at some region of parameter space. This means that the inequality is strong enough to point out problems in various approximation schemes, which may be the first step leading either to improving or discarding those schemes. Finally, we consider a number of experimental and numerical studies that yield exponents in violation of the inequality [8, 9, 10, 20, 21, 22, 23] and suggest measurements that would clarify the origin of the problem.

Many interesting dynamical physical systems may be described in terms of some physical field,  $\phi(\mathbf{r}, t)$  driven by a "noise" field,  $\eta(\mathbf{r}, t)$ . The list of systems, described by generic Langevin field equations, includes models of critical dynamics [1], growth models of the KPZ family [2] and its many variants [4, 5, 6], noise driven Navier Stokes [24] etc.. Strictly speaking, the physical field given as a function of time depends not only on the noise field at earlier times but also on initial conditions. The dependence on initial conditions decays, however, in time, and we are left with an implicit relation between the Fourier transform of the field and the Fourier transform of the noise,

$$\phi(\mathbf{q}, \omega) = \phi\{\mathbf{q}, \omega; \eta(\mathbf{l}, \sigma)\}, \quad (1)$$

where  $\eta(\mathbf{l}, \sigma)$  is a Gaussian random field with  $\langle \eta(\mathbf{l}, \sigma) \rangle =$

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0, and

$$\langle \eta(\mathbf{l}, \sigma) \eta(\mathbf{m}, \varsigma) \rangle = 2D_0(l, \sigma) \delta(\mathbf{l} + \mathbf{m}) \delta(\sigma + \varsigma). \quad (2)$$

We use the traditional convention and take the physical units of the noise to be equal to the physical units of the physical field  $\phi$  over time. The loss of initial condition information is characteristic of the steady state and thus the various averages we will consider in the following, which are averages over the noise are also steady state averages, traditionally studied in the literature [1].

We are interested in the response function  $G(q, \omega)$ , to be defined by

$$\left\langle \frac{\delta \phi(\mathbf{q}, \omega)}{\delta \eta(\mathbf{p}, \sigma)} \right\rangle \equiv G(q, \omega) \delta(\mathbf{q} - \mathbf{p}) \delta(\omega - \sigma), \quad (3)$$

and in the correlation function  $\Phi(q, \omega)$ , which is defined by

$$\langle \phi(\mathbf{q}, \omega) \phi(-\mathbf{p}, -\sigma) \rangle \equiv \Phi(q, \omega) \delta(\mathbf{q} - \mathbf{p}) \delta(\omega - \sigma). \quad (4)$$

Because of the Gaussian character of the noise, using integration by parts the response function can be written also as

$$G(q, \omega) \delta(\mathbf{q} - \mathbf{p}) \delta(\omega - \sigma) = \langle \phi(\mathbf{q}, \omega) \eta(-\mathbf{p}, -\sigma) \rangle / 2D_0(q, \omega). \quad (5)$$

Note, that if we **define** the response function by the right hand side of equation (5) (which still involves a non-trivial correlation function) the rest of the derivation follows, **even** if the distribution of the noise is **not** Gaussian.

The average  $\langle \chi(\mathbf{q}, \omega) \psi(-\mathbf{p}, -\sigma) \rangle$  can be viewed as a scalar product of  $\chi(\mathbf{q}, \omega)$  and  $\psi(\mathbf{p}, \sigma)$ , since it has all the properties required of a scalar product. Using the Schwartz inequality we find

$$\begin{aligned} & |G(q, \omega)| \delta(\mathbf{q} - \mathbf{p}) \delta(\omega - \sigma) \\ & \leq \sqrt{\Phi(q, \omega) 2D_0(q, \omega)} \delta(\mathbf{q} - \mathbf{p}) \delta(\omega - \sigma) / 2D_0(q, \omega). \end{aligned} \quad (6)$$

Integrating over  $\mathbf{p}$  and  $\sigma$  and squaring both sides leads to

$$2G(q, \omega) G(-q, -\omega) D_0(q, \omega) \leq \Phi(q, \omega). \quad (7)$$

The above is a general exact inequality, relating the response function, as defined by the right hand side of equation (5) and the correlation function. To turn that into an exponent inequality, let the equal time correlation,

$$\Lambda(q) = \int_{-\infty}^{\infty} d\omega \Phi(q, \omega), \quad (8)$$

be proportional to  $q^{-\Gamma}$  for small  $q$  and let the response exponent  $\bar{z}$  characterize the small  $q$  behavior of the response function

$$G(q, 0) \propto q^{-\bar{z}}, \quad (9)$$

The characteristic frequency,  $\omega(q)$ , associated with the decay in time of the correlation is given in terms of the dynamic exponent  $z$  as [1]

$$\omega^{-1}(q) = \pi \Phi(q, 0) / \Lambda(q). \quad (10)$$

and is proportional to  $q^z$ . (The use of a small  $q$  power law behavior of  $\Lambda$ ,  $G(q, 0)$  and  $\omega$  is not a restriction at all as any dependence that is not a power law at small  $q$  can be described as a power law with a power which is either zero or  $\pm$  infinity.)

We will concentrate in the following on bare spectral functions  $D_0(q, \omega)$  (the noise correlator in Eq. (2)) that for small  $q$  and  $\omega$  have the form

$$D_0(q, \omega) = Bq^{-2\sigma} \text{ with } \sigma \geq 0 \text{ for small } q \text{ and } \omega. \quad (11)$$

The above form is rich enough to make our point. Still, the discussion of more general cases is straightforward.

The required exponent inequality is obtained now by setting  $\omega = 0$  in Eq. (7),

$$2\bar{z} + 2\sigma \leq \Gamma + z. \quad (12)$$

The response exponent  $\bar{z}$  is usually related to other exponents characterizing the system. For example in Model A of Hohenberg and Halperin [1]  $\bar{z} = \Gamma$ . The latter is an exact relation, which leads to the correct but not very exciting inequality  $\Gamma \leq z$ . In growth models of the KPZ family, we have, instead, the relation,  $z = \bar{z}$ , [25, 26]. It does not seem, however, very likely that the two options mentioned above,  $\bar{z} = \Gamma$  and  $\bar{z} = z$  exhaust all the interesting possibilities. Independent determination of  $\bar{z}$  in other systems might be thus of interest.

Note that while inequality (12) is rigorous, we will use from now on additional scaling relations, relating the exponents, which are most probably correct and widely used but have never been rigorously proven. Using the relation  $\bar{z} = z$ , the general inequality above reduces to the simpler

$$z \leq \Gamma - 2\sigma \leq \Gamma. \quad (13)$$

It is well known that for members of the KPZ family, there is a lower critical dimension above which appears a weak coupling solution on top of the strong coupling solution, which is the only possibility at lower dimensions [6]. The above inequality is correct for any set of exponents, whether strong or weak coupling and in fact for weak coupling exponents, we expect the leftmost inequality to hold as an equality.

To demonstrate the usefulness of the exponent inequality, we use it to test various theoretical approaches to the study of the KPZ family. We concentrate on the non-local KPZ, where we expect that there is a good chance that inherent problems in various theoretical approaches will show up clearly in violations of the inequality. We find that that is indeed the case and all the methods

tested apart from the SCE [17, 18, 19] violate the inequality.

The Non-local KPZ (NKPZ) equation has been introduced in Ref. [4] to account for the non-local hydrodynamic interactions in a deposition of colloidal particles in a fluid. It was later generalized to spatial correlated noise ( $\sigma \neq 0$ ) in Ref. [5]. The NKPZ equation for the height function,  $h(\mathbf{r}, t)$  is given by

$$\partial_t h(\mathbf{r}, t) = \nu \nabla^2 h + \int d\mathbf{r}' g(\mathbf{r}') \nabla h(\mathbf{r} + \mathbf{r}', t) \cdot \nabla h(\mathbf{r} - \mathbf{r}', t) + \eta, \quad (14)$$

where the kernel  $g(\vec{r})$  has a short range part,  $\lambda_0 \delta^d(\vec{r})$ , and a long-range part  $\sim \lambda_\rho r^{\rho-d}$ . In Fourier space,  $\hat{g}(q) = \lambda_0 + \lambda_\rho q^{-\rho}$ . In order to avoid an extremely, complicated phase diagram, we discuss here only the case  $\lambda_0 = 0$ . The noise has zero mean, but is allowed to have spatially long range correlations, given by equation (11). (The roughness exponent,  $\alpha$ , we are interested in, is related in KPZ to the exponent  $\Gamma$  by  $2\alpha = \Gamma - d$ , where  $d$  is the dimensionality of the system.) The strong coupling solution found by the Dynamical Renormalization Group (DRG) is [4]

$$z_{DRG} = 2 + \frac{(d-2-2\rho)(d-2-3\rho)}{(3+2^{-\rho})d-6-9\rho}. \quad (15)$$

(Because of the extra scaling relation between  $\alpha$  and  $z$  [4, 5], namely  $\alpha + z = 2 - \rho$ , there is only one independent exponent in NKPZ.)

The above result violates the inequality (13) over a whole range of parameters defined by  $\Gamma_{DRG} - z_{DRG} - 2\sigma < 0$  or explicitly given for the weaker inequality, with  $\sigma = 0$ , by

$$\frac{d-2}{2} \leq \rho \leq \rho_0(d) \quad \text{for } d < d_0, \quad (16)$$

$$\rho_0(d) \leq \rho \leq \frac{d-2}{2} \quad \text{for } d > d_0,$$

with  $d_0 \simeq 3.395$  and  $\rho_0(d) = \frac{d-2}{3} + \frac{W\left(\frac{1}{9}2^{\frac{2-d}{3}}d \ln 2\right)}{\ln 2}$ , where  $W(x)$  is the Lambert function. The shaded region in Fig. 1 is the region in the  $(d, \rho)$  plane where the weaker inequality (i.e. Eq. (13) with  $\sigma = 0$ ) is violated.

In Fig. 2 we give the solution for the dynamic exponent  $z$  for  $d = 1, 2, 3$  by an improved mode-coupling approach originally due to Colaioni and Moore for the local case [27], which was later applied by Hu and Tang [28] to the non-local case. The shaded area bounded by the line  $z = (d + 4 - 2\rho)/3$  marks the region where the inequality is violated.

Note, that in both of the above examples the boundary of the violation is  $\sigma$  independent. Namely, we have used the weaker inequality with  $\sigma = 0$ ,  $\Gamma \geq z$ . The use of the stronger inequality (Eq. (13) with positive  $\sigma$ ) puts more stringent bounds and therefore, the actual regions where violations occur are larger, depending on the value of  $\sigma$ .

Next, we check for violations of the inequality in results obtained by Tang and Ma [29] generalizing the Flory-type Scaling Approach (SA) of Hentschel and Family [30]

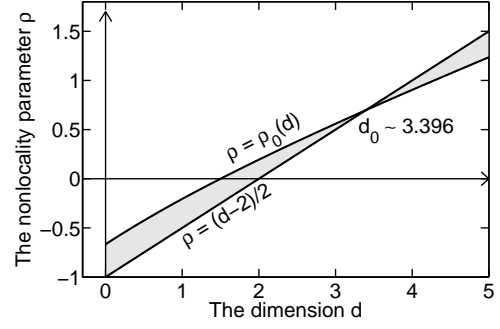


FIG. 1: Violation of the response-correlation inequality (13) with  $\sigma = 0$  by the DRG method derived in Refs. [4, 5] occurs in the shaded area enclosed by the curves  $\rho = (d-2)/2$  and  $\rho = \rho_0(d)$  in the phase diagram.

to the non-local case. The strong-coupling dynamic exponent obtained by using this method is

$$z_{SA} = \frac{(2-\rho)(d+2-2\sigma)}{d+3-2\sigma}. \quad (17)$$

It turns out that this solution violates the inequality in a whole region of parameter space defined by

$$\frac{d}{2} < \sigma < \frac{d+1+\rho}{2} \quad \text{for } \rho > -1, \quad (18)$$

$$\frac{d+1+\rho}{2} < \sigma < \frac{d}{2} \quad \text{for } \rho < -1. \quad (19)$$

These results are presented graphically in Fig. 3 below.

Last we test the results for NKPZ of the Schwartz-Edwards Self-Consistent Expansion (SCE) [17, 18]. This method predicts a whole zoo of possible phases summarized in Table I in Ref. [19]. It turns out that these results are not only consistent with the exact one-dimensional result [12] but the exponents in all the phases are consistent with our stronger inequality over the whole parameter space and all dimensionalities. In Fig. 4 below we give the SCE solution for the dynamic exponent  $z$  for  $d = 1, 2, 3$  for uncorrelated noise ( $\sigma = 0$ ) (but as stated above the inequality holds also for  $\sigma > 0$ ).

Our inequality, is shown thus to be a useful tool in detecting shortcomings of various analytical methods applied to the NKPZ family and consequently suggest that results obtained by those methods should be suspected even in cases where the inequality is not violated, at least until the origin of violation of the inequality is understood.

We have reviewed the literature and found some numerical and experimental results that violate the inequality (13). We have found violations for crack surfaces in wood [8] and in mortar [9, 10], where the dynamic exponent  $z$  and the roughness exponent  $\alpha$  are reported. In

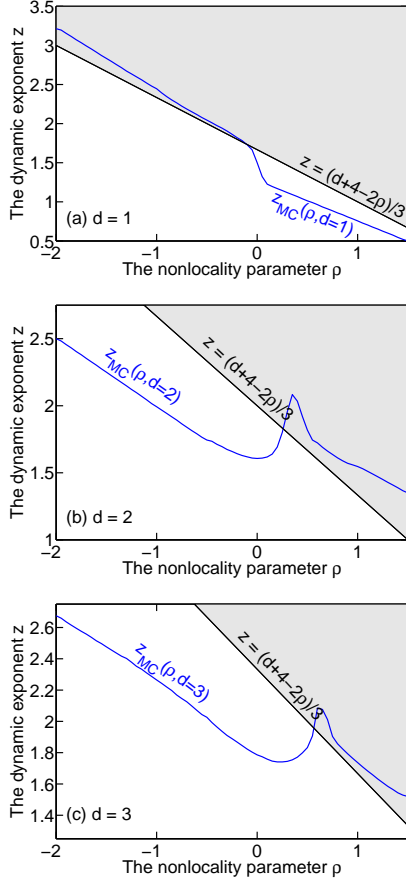


FIG. 2: The Dynamic exponent  $z$  as a function of the nonlocality parameter  $\rho$  using the mode coupling result (blue line - reproduced from Ref. [28]). The part of the blue line within the shaded region is excluded by the inequality (13) with  $\sigma = 0$ .

addition, we have found violations in a number of surface growth systems, including interface roughness in Fe-Cr superlattices [20], growth front roughening in silicon nitride films by plasma enhanced chemical vapor deposition [21], interface roughening in wrinkly metal [22] and effects of shadowing on growth processes [23], where all three exponents  $\alpha, \beta$  and  $z$  are measured independently. The simplest explanation for this violation could be just due to one or more of the numerous artifacts in the methods used to extract the exponents. Possible artifacts are discussed thoroughly in Refs. [31, 32, 33]. An alternate explanation for the problematic exponents could be that the time involved in the measurement is too short for the system to reach steady state while the roughness exponent  $\alpha$ , should be measured in steady state.

There are, however, two more alternatives. Experiments and numerical simulation often produce non-steady-state quantities. For example,

$$W(r, t) = \langle [h(\mathbf{r}, t) - h(0, t)]^2 \rangle, \quad (20)$$

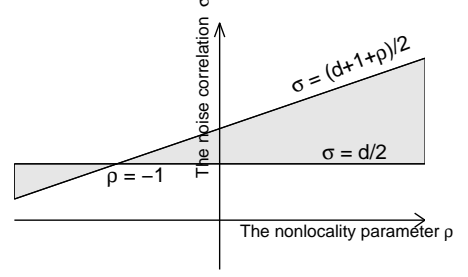


FIG. 3: Violation of the response-correlation inequality (13) by the SA method derived in Ref. [29] occurs in the shaded area enclosed by the curves  $\rho = -1$  and  $\sigma = (d + 1 + \rho)/2$  in the phase diagram.

with  $h(\mathbf{r}, 0) = 0$ . This tends to a steady state quantity only as time tends to infinity. Any  $q$  dependent characteristic time obtained from Eq. (20), can be related to the steady state exponent  $z$  only if a single characteristic time exists for each length scale. In the presence of more than one characteristic time-scale for each length scale, the relation between steady-state and non-steady-state dynamic exponents is not obvious and our inequality relating  $z$  to  $\alpha$  cannot be invoked. Moreover, even for steady-state we have to recall that our inequality (13) is based on the relation  $z = \bar{z}$ . For the KPZ family we understand the origin of that relation but for other systems this might not be necessarily true (as for example in critical dynamics [1]). Therefore only the more general inequality (12), involving  $z$  and  $\bar{z}$ , should be used. To test the exponents obtained by one method or another, one would need to obtain not only the time dependent steady state correlation function,

$$\Phi(\mathbf{r}, t) = \langle [h(\mathbf{r}, t) - h(0, 0)]^2 \rangle, \quad (21)$$

but also the corresponding response function,

$$G(\mathbf{r}, t) = \left\langle \frac{\delta h(\mathbf{r}, t)}{\delta \eta(0, 0)} \right\rangle. \quad (22)$$

The fact that the scaling relation  $z = \alpha/\beta$  is violated in the surface growth systems mentioned above might perhaps indicate the existence of more than a single characteristic time scale, but we cannot be confident of that. In fact it might also be an indication that steady state has not been reached. In order to clarify this issue we suggest that measurements such as those described in Refs. [20, 21, 22, 23], would be complemented by steady state measurements of time dependent correlation functions and response functions given by equations (21) and (22). This does not require much additional effort, since in order to obtain  $\alpha$ , the system has anyhow to be studied for times long enough to reach steady state. The independent extraction of the exponents  $z$  and  $\bar{z}$  will enable

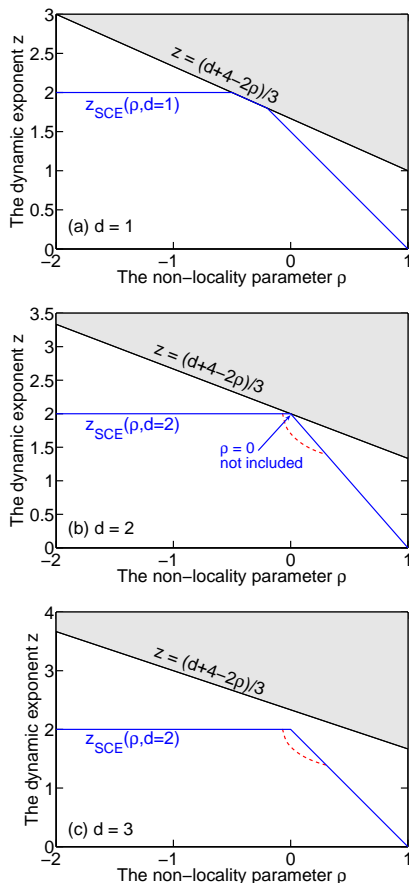


FIG. 4: (color online) The dynamic exponent  $z$  as a function of the non-locality parameter  $\rho$  for uncorrelated noise ( $\sigma = 0$ ) in  $d = 1, 2, 3$  dimensions resulting from the Self-Consistent Expansion [19]. For  $d = 1$  the solution saturates the inequality for  $-1/2 < \rho < -1/5$ . Note that for  $d = 2, 3$  two phases are possible in some range of the parameters as seen from the fact that  $z$  is a multivalued function of  $\rho$  (solid (blue) line weak coupling solution, and dashed (red) line strong coupling solution). Also note that for  $d = 2$  and  $\rho = 0$  there is only one phase as the solid (blue) line does not exist for  $\rho = 0$ .

to check (a) whether the non steady state  $z$  obtained for example from the correlation (20) and the steady state  $z$  are equal and (b) if  $z$  equals  $\bar{z}$ . This and the use of the more general inequality (12) will put the quality of the data to a much more stringent test. Violation of inequality (12) will then certainly indicate that the exponents obtained are inaccurate for one reason or another.

To summarize, in this paper we showed how to generalize the Schwartz-Soffer inequality derived originally for quenched random systems [16] to dynamical stochastic systems. We show that the inequality, which involves the correlation and the response functions can be translated into a simple inequality for the scaling exponents  $\Gamma$ ,  $z$  and the noise correlation exponent  $\sigma$ , in cases like the KPZ family, where  $z = \bar{z}$ . Although being extremely simple, this inequality can be quite powerful when exam-

ining analytical, numerical and experimental results.

To demonstrate the utility of the inequality, we reviewed analytical results for the non-local KPZ model, obtained by using four different methods: Dynamical Renormalization Group, Mode-Coupling, Scaling-Approach and the Self-Consistent Expansion. Interestingly, the first three methods yield results which contradict the inequality for a whole set of parameters, while the Self-Consistent Expansion is the only one which never violates it. This has an important implication on the choice of analytical tools when dealing with such stochastic models.

Last, we reviewed a set of numerical and experimental results coming from systems of surface growth and post-mortem analysis of crack surfaces. We suggest to measure the steady state response and correlation functions, which for surface growth systems, does not seem to require much additional effort. In case the more general inequality is violated it will become clear that the quality of the data and/or the methods of obtaining the exponents from it are not satisfactory.

An interesting open question is whether there exist interesting cases where the response exponent  $\bar{z}$  is not equal to  $\Gamma$  or to the dynamic exponent  $z$ .

Hopefully, the simplicity yet strength of this result will also motivate researchers to explore the usefulness of rigorous inequalities and derive improved ones alongside the more popular chase of approximated equalities.

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